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Ideals in Ortholattices, Bell Inequalities, and Simultaneously Definite Properties

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The notion of quasiboolean algebras (Bell and Clifton, 1995) is compared with related notions of semiprime ideals, commutator ideals, partial compatibility, joint distributions of observables, and Bell inequalities on orthomodular lattices. Some consequences of characterizations of simultaneously definite properties are derived.

1. INTRODUCTION

Recently, Bell and Clifton (1995) introduced the notion of quasi-Boolean algebras and used them as a unifying element for different approaches to a characterization of simultaneously definite properties in quantum mechanics.

One physical motivation for this approach is a result first proven by Kochen and Specker (1967) (see also Specker, 1960; Zierler and Schlessinger, 1965; Bell, 1966; Mermin, 1993) stating the impossibility to "complete" quantum physics by the introduction of noncontextual hidden parameter models. Such a possible "completion" had been suggested, though in not very concrete terms, by Einstein, Podolsky, and Rosen (1935) (EPR). These authors speculated that "elements of physical reality" exist irrespective of whether they are actually measured or not. Moreover, EPR conjectured, the quantum formalism can be "embedded" into a larger theoretical framework which would reproduce the quantum-theoretic results but would otherwise be classical and deterministic from an algebraic and logical point of view.

A proper formalization of the term "element of physical reality" suggested by EPR can be given in terms of two-valued states or valuations, which can take on only one of two values 0 and 1 and which are interpretable

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as the classical logical truth assignments *false* and *true*, respectively. Kochen and Specker's (1967) results state that for quantum systems representable by Hilbert spaces of dimension higher than two, there does not exist any such valuation $f: L \to \{0, 1\}$ on the set of closed linear subspaces L interpretable as quantum mechanical propositions preserving the lattice operations and the orthocomplement, even if these lattice operations are carried out among commuting elements.[3]

The Kochen and Specker result, it is commonly believed, is directed against the noncontextual hidden parameter program envisaged by EPR. Indeed, if one takes into account the entire logic spanned by Hilbert space (of dimension larger than two) and if one considers all states thereon, any truth value assignment to quantum propositions prior to the actual measurement yields a contradiction. One "fallback" option we shall consider here is a restriction to a subset of all possible states, corresponding to specifications of physical preparation procedures. It is then possible to recover consistent truth value assignments and therefore valuations for a subclass of quantum mechanical propositions. In this way one is naturally led to the notion of quasiboolean algebras.

The aim of the present paper is to put these ideas into relations with related results previously obtained in the lattice theory and quantum logic approach to quantum mechanics. In particular, we are interested in semiprime ideals, Boolean quotients, commutators, partial compatibility, joint distributions, Bell inequalities, and hidden variables.

In Rav (1989) the ring-theoretic concept of semiprime ideal is appropriately defined for lattices. It is proved that an ideal I of a lattice L is semiprime iff I is the kernel of some homomorphism of L onto a distributive lattice with 0. The theory of semiprime ideals is developed there without assuming the axiom of choice and it is proved that the Ultrafilter Principle[4] is equivalent to the statement that every semiprime ideal is representable as an intersection of prime ideals.

In Beran (1987), distributivity of a finitely generated orthomodular lattice is characterized using the concept of a semiprime ideal. A generalization of these results can be found in Chevalier (1988), where it is proved that an ideal in an orthomodular lattice is semiprime iff it contains the commutator ideal.

[3] Nonpreservation of lattice operations among noncommuting propositions is quite evident, given the nondistributive structure of quantum logics.

[4] The Ultrafilter Theorem for Boolean algebras says that every Boolean algebra contains a maximal filter (an ultrafilter, equivalently; recall that an ultrafilter in a Boolean algebra B is a proper filter F such that for any $x \in B$, either $x \in F$ or $x' \in F$). Unlike the case of commutative rings, the Axiom of Choice is strictly stronger than the Ultrafilter Theorem for Boolean algebras (Banaschewski, 1983).

Irreducible orthomodular lattices with Boolean quotients are studied in D'Andrea and Pulmannová (1995), where a nontrivial example of an irreducible orthomodular lattice is found, all proper quotients of which are Boolean algebras.

Some Bell-type inequalities in orthomodular lattices and their relations to subadditivity of states, commutators, and Boolean quotients are studied in Pulmannová and Majerník (1992), Pulmannová (1994), and D'Andrea and Pulmannová (1995) and a further development of these ideas can be found in Dvurečenskij and Länger (1995a, b) and Pykacz and Santos (1995).

2. SEMIPRIME IDEALS AND QUASIDISTRIBUTIVE LATTICES

In the theory of commutative rings, the following result is known as Krull's Lemma (Banaschewski, 1983): The Axiom of Choice is equivalent, in Zermelo–Frankel set theory, to the condition that any proper ideal in a commutative ring with unit is contained in a maximal ideal, which in turn is equivalent to the formally simpler condition that any nontrivial commutative ring with unit contains a maximal ideal. Another variant of Krull's Lemma says that the Boolean Ultrafilter theorem is equivalent to the condition that every nontrivial commutative ring with unit contains a prime ideal. There have been several attempts to find an analogue of Krull's Lemma in the theory of distributive lattices (Banaschewski, 1983). Recall that an ideal I of a commutative ring R with unit is called semiprime whenever $a^n \in I$ (n is a positive integer) entails $a \in I$. According to a well-known result by Krull (1929), using the well-ordering theorem, every semiprime ideal is the intersection of all prime ideals that contain it [see Rav (1977) for a proof using only the Ultrafilter Principle]. In the following definition, an appropriate analogue of the notion of a semiprime ideal for lattices is given (Rav, 1989).

Recall that an ideal in a lattice L is a subset I of L such that $a \in I$ and $b \leq a$ imply $b \in I$, and $a, b \in I$ implies $a \vee b \in I$. The definition of a filter is dual, that is, a subset F of L is a filter if $a \in F$ and $b \geq a$ imply $b \in F$ and $a, b \in F$ implies $a \wedge b \in F$. An ideal (filter) is proper if it is not equal to the whole L, and a proper ideal (filter) is maximal if there is no bigger proper ideal (filter).

Definition 2.1. An ideal I of a lattice L is called *semiprime* if for every $x, y, z \in L$, whenever $x \wedge y \in I$ and $x \wedge z \in I$, then $x \wedge (y \vee z) \in I$. Dually, a filter F is *semiprime* if $x \vee y \in F$ and $x \vee z \in F$ imply that $x \vee (y \wedge z) \in F$.

In a distributive lattice, every ideal and every filter is semiprime. Recall that an ideal I is called *prime* if $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$. It is easy to see that every prime ideal is semiprime, and consequently any nonempty intersection of prime ideals or filters is semiprime.

The Ultrafilter Principle (in the form of the Boolean Prime Ideal Theorem) says that every Boolean algebra contains a prime ideal. Rav (1989) proves that the Ultrafilter Principle is equivalent to the statement that every semiprime ideal in a lattice is representable as an intersection of prime ideals (a dual result holds for filters).

An ideal I is *principal* if $I = (a] = \{b \in L: b \leq a\}$. We note that for principal ideals, the notion of a semiprime ideal coincides with the notion of 0-distributivity due to Varlet (1968). According to Varlet (1968), a lattice with 0 is *0-distributive* if $x \wedge y = 0$ and $x \wedge z = 0$ implies that $x \wedge (y \vee z) = 0$. In Rav (1989) a 0-distributive lattice is called semiprime (i.e., the zero ideal (0] is semiprime). Dually, a lattice with 1 is called *dual-semiprime* if the unit filter [1) is semiprime. A bounded[5] lattice is called *bi-semiprime* if it is both semiprime and dual semiprime.

Recall that a binary relation R in a lattice is a *congruence* if R is an equivalence relation preserving lattice operations, i.e., aRa_1 and bRb_1 imply $a \vee bRa_1 \vee b_1$ and $a \wedge bRa_1 \wedge b_1$. A mapping $h: L_1 \to L_2$, where L_i, $i = 1, 2$, are lattices, is a *homomorphism* if it preserves the lattice operations, i.e., $h(a \vee b) = h(a) \vee h(b)$ and $h(a \wedge b) = h(a) \wedge h(b)$. The *kernel* of a homomorphism is the set $\{a \in L: h(a) = 0\}$. The kernel of any homomorphism is an ideal, but not every ideal gives rise to a homomorphism in general. Let R be a congruence on a lattice L. For $a \in L$, let $\bar{a}$ denote the equivalence class with respect to R to which a belongs. The set of all equivalence classes, denoted by L/R, is a lattice called a *quotient* of L. The mapping $a \mapsto \bar{a}$ assigning to every element $a \in L$ its corresponding equivalence class $\bar{a}$ in L/R is a surjective homomorphism (called also the canonical epimorphism).

The main theorem in Rav (1989) is the following.

Theorem 2.2. Let L be a lattice and I an ideal in L. Then the following conditions are equivalent:

1. I is semiprime.
2. I is the kernel of some homomorphism onto a distributive lattice with zero.
3. I is the kernel of a homomorphism of L onto a semiprime lattice.

In Bell and Clifton (1995), the notion of an I-quasidistributive lattice (strongly I-quasidistributive lattice) is introduced as follows. As usual, the symbol 2 denotes the two-element Boolean algebra $\{0, 1\}$.

Definition 2.3. Let L be a lattice and I an ideal in L. The lattice L is called *I-quasidistributive* if (one of) the following equivalent conditions are satisfied.

[5] A lattice is bounded if it has a smallest element 0 and a greatest element 1.

1. $\text{Rad}(L) \subseteq I$, where $\text{Rad}(L)$ denotes the intersection of the family of all prime ideals in L.
2. Any $x \notin I$ is contained in a prime filter.
3. For any $x \notin I$ there is a homomorphism h: $L \to 2$ such that $h(x) = 1$.
4. There is a Boolean algebra B and a lattice homomorphism f: $L \to B$ such that $f^{-1}(0) \subseteq I$.

Definition 2.4. A lattice L with an ideal I is called *strongly I-quasidistributive* if the following equivalent conditions are satisfied.

1. I is the intersection of a (nonempty) family of prime ideals.
2. Any $x \notin I$ is contained in a prime filter F such that $F \cap I = \emptyset$.
3. For any $x \notin I$ there is a homomorphism h: $L \to 2$ such that $h(x) = 1$ and $I \subseteq h^{-1}(0)$.
4. There is a Boolean algebra B and a lattice homomorphism f: $L \to B$ such that $I = f^{-1}(0)$.
5. Every I-maximal filter is prime. (A filter F is said to be I-maximal if it is maximal with respect to the property of disjointness from I.)

Proposition 2.5. Let L be a lattice, I an ideal in L. The following conditions are equivalent:

1. I is semiprime iff L is strongly I-quasidistributive.
2. The Ultrafilter Principle (UP) holds.

Proof. By Rav (1989, Theorem 4.2), every semiprime ideal of a lattice is representable as an intersection of prime ideals iff UP holds. By (1) in the definition of strongly I-quasidistributive lattice, the result follows. ■

3. IDEALS IN ORTHOLATTICES AND ORTHOMODULAR LATTICES

Recall that an *ortholattice* (OL) is a (bounded) lattice with orthocomplementation, i.e., a unary operation $'$: $L \to L$ such that, for all $a, b \in L$, (i) $a \leq b \Rightarrow b' \leq a'$, (ii) $(a')' = a$, (iii) $a \vee a' = 1$ (dually, $a \wedge a' = 0$). An ortholattice becomes an *orthomodular lattice* (OML) if the orthomodular law

$$a \leq b \Rightarrow b = a \vee (a' \wedge b)$$

holds.

A congruence in an ortholattice should also preserve the orthocomplementation, i.e., $aRb \Rightarrow a'Rb'$. In an orthomodular lattice every lattice congruence is a congruence (see, e.g., Beran, 1987). Similarly, a homomorphism of an ortholattice should preserve orthocomplements and map the unit element to the unit element.

An ideal I in an ortholattice L is called an *orthomodular ideal* (or a p-ideal) if

$$a \in I, \quad b \in L \Rightarrow (a \vee b) \wedge b' \in I$$

Dually, a filter F in L is called an *orthomodular filter* (or a p-filter) if

$$a \in F, \quad b \in L \Rightarrow (a \wedge b) \vee b' \in F$$

Lemma 3.1. Let F be a subset of an OL L. The following statements are equivalent:

1. F has the following properties: (i) $1 \in F$, (ii) $x \in F$ and $x' \vee y \in F$ imply $y \in F$.
2. F is an orthomodular filter.

Proof. (1) $\Rightarrow$ (2): We prove first that F with properties (i) and (ii) is a filter. If $x \in F$, $x \leq y$, then $x \wedge y = x \in F$ and $(x \wedge y)' \vee y = x' \vee y' \vee y = 1 \in F$ implies $y \in F$. If $x, y \in F$, then $x' \vee y' \vee (x \wedge y) = 1 \in F$ implies $y' \vee (x \wedge y) \in F$, which in turn implies $x \wedge y \in F$. To prove that F is an orthomodular filter, for $x \in F$, $y \in L$ define $z = (x \wedge y) \vee y'$. Then $x' \vee z = x' \vee y' \vee (x \wedge y) = 1 \in F$ implies $z \in F$.

(2) $\Rightarrow$ (1): Assume $x \in F$ and $x' \vee y \in F$. Then $(x' \vee y \vee y') \wedge y = y \in F$. ■

According to Cignoli (1978), a subset F of an ortholattice L with properties (i) and (ii) is called a *deductive system* (see also Kalmbach, 1983).

A *Boolean deductive system* is a deductive system F such that the relation $\{(x, y) \in L^2: (x' \vee y) \wedge (y' \vee x) \in F\}$ is a congruence relation.

Define, for $x, y \in L$,

$$x \rightarrow y = (x' \wedge y) \vee (x' \wedge y') \vee (x \wedge (x' \vee y'))$$

An *orthomodular deductive system* is a deductive system such that the relation $\{(x, y) \in L^2: (x \rightarrow y) \wedge (y \rightarrow x) \in F\}$ is a congruence relation on L.

The following statement characterizes those ideals in an ortholattice which are kernels of homomorphisms onto Boolean algebras (Kalmbach, 1983).

Proposition 3.2. Let L be an ortholattice and let I be a proper ideal of L. The following statements are equivalent:

1. There exists a homomorphism h from L onto a Boolean algebra with $h^{-1}(0) = I$.
2. I is the intersection of prime ideals of L.
3. $\{x' \in L: x \in I\}$ is a Boolean deductive system.

In Bell and Clifton (1995) a quasidistributive ortholattice is called a quasi-Boolean algebra. As a corollary of Proposition 3.2 we obtain that an ortholattice L is strongly I-quasi-Boolean if and only if the set $\{x' \in L: x \in I\}$ is a Boolean deductive system. Moreover, L is strongly I-quasi-Boolean if and only if I is the intersection of prime ideals, consequently I is semiprime. The next proposition shows that the converse statement is also true.

Proposition 3.3. If I is a semiprime ideal of an ortholattice L, then the set $F = \{x' \in L: x \in I\}$ is a Boolean deductive system.

Proof. We have to prove that $x \equiv y$ iff $(x \vee y') \wedge (y \vee x') \in F$ is a congruence relation. Recall that a reflexive binary relation Θ on a lattice is a congruence relation iff:

(a) $x \equiv y(\Theta)$ iff $x \wedge y \equiv x \vee y(\Theta)$.

(b) $x \leq y \leq z$, $x \equiv y(\Theta)$, and $y \equiv z(\Theta)$ imply $x \equiv z(\Theta)$.

(c) $x \leq y$ and $x \equiv y(\Theta)$ imply $x \wedge t \equiv y \wedge t(\theta)$ and $x \vee t \equiv y \vee t(\Theta)$.

To prove (i), observe that $x \wedge y \equiv x \vee y$ iff $x \wedge y \vee x' \wedge y' \in F$, and $x \wedge y \vee x' \wedge y' \leq (x \vee y') \wedge (x' \vee y)$ implies that $x \equiv y$. Conversely, $x \equiv y$ implies $x \wedge y' \vee x' \wedge y \in I$. Then $x \wedge y' \in I$ and $0 = x \wedge x' \in I$ imply, since I is semiprime, that $x \wedge (y' \vee x') \in I$, and similarly we prove that $y \wedge (y' \vee x') \in I$. Using once again the semiprime property of I, we obtain that $(x \vee y) \wedge (x' \vee y') \in I$, hence $x \vee y \equiv x \wedge y$.

To prove (ii), observe that $x \leq y \leq z$, and $x \equiv y$, $y \equiv z$ imply $x' \wedge y \in I$ and $y' \wedge z \in I$. Then $x' \wedge z \wedge y' = z \wedge y' \in I$ and $x' \wedge z \wedge y = x' \wedge y \in I$, and since I is semiprime, this yields $x' \wedge z \wedge (y \vee y') \in I$, hence $x \equiv z$.

To prove (iii), observe that $x \leq y$ and $x \equiv y$ imply $x' \wedge y \in I$. Then $x' \wedge y \wedge t \leq x' \wedge y \in I$ and $0 = t' \wedge y \wedge t \in I$ imply $(x' \vee t') \wedge t \wedge y \in I$, hence $x \wedge t \equiv y \wedge t$. Similarly, $x' \wedge t' \wedge y \leq x' \wedge y \in I$ and $x' \wedge y \wedge t' = 0 \in I$ imply that $x' \wedge t' \wedge (y \vee t) \in I$. Hence $x \vee t \equiv y \vee t$. ■

Corollary 3.4. In an ortholattice L the following statements are equivalent:

1. I is a semiprime ideal of L.
2. L is strongly I-quasi-Boolean.
3. I is the intersection of prime ideals.
4. $\{x \in L: x' \in I\}$ is a Boolean deductive system.

Corollary 3.4 extends Krull's result to ortholattices and semiprime ideals of them. For orthomodular lattices this result was proved in Chevalier (1988) (see Proposition 3.6 below).

The following proposition characterizes those ideas (filters) of an ortholattice which are kernels of homomorphisms onto an orthomodular lattice. For the proof see Kalmbach (1983).

Proposition 3.5. Let L be an ortholattice, I a proper subset of L, and $F = \{x \in L: x' \in I\}$. The following statements are equivalent:

1. There exists a homomorphism h from L onto an orthomodular lattice with $h^{-1}(0) = I$.
2. F is an orthomodular deductive system.

It is easy to check that a homomorphic image of an orthomodular lattice is an orthomodular lattice [indeed, let $h(a) \leq h(b)$. Orthomodularity gives $a \vee b = a \vee (a' \wedge (a \vee b))$, which implies $h(b) = h(a) \vee h(a)' \wedge h(b)$]. Let I be an ideal in an orthomodular lattice L. It is well known that the following statements are equivalent:

1. I is an orthomodular ideal.
2. Relation aRb iff $a\Delta b = (a \vee b) \wedge (a' \vee b') \in I$ is a congruence.
3. I is the kernel of a homomorphism of L.

Let L be an orthomodular lattice (OML, for short). For $a, b \in L$, the element

$$\overline{com}(a, b) = (a \vee b) \wedge (a' \vee b) \wedge (a \vee b') \wedge (a' \vee b')$$

is called the *upper commutator of a, b.* The lower commutator is defined dually:

$$\underline{com}(a, b) = \overline{com}(a, b)'$$

Any ideal in an OML which contains the ideal I_c generated by the upper commutators is orthomodular and, for an orthomodular ideal I of L, the quotient L/I is Boolean iff $I_c \subseteq I$. The ideal I_c is called the *commutator ideal* (Marsden, 1970). The following results were obtained in Chevalier (1988):

Proposition 3.6. Let I be an ideal of an OML L. The following statements are equivalent:

1. I is semiprime.
2. I satisfies the condition $a \wedge b \in I$ and $a \wedge b' \in I$ imply $a \in I$.
3. I contains the commutator ideal.

Condition 3 implies that every semiprime ideal is orthomodular. Taking into account that an ideal I of an OML is prime iff I is orthomodular and $L/I = 2$, we obtain that an ideal I is semiprime iff it is the intersection of all prime ideals that contain it. Indeed, we clearly have $I \subset \bigcap\{J \text{ prime}: I \subset J\}$, and if $a \notin I$, $a \in J$ for any prime ideal J which contains I, then in the Boolean algebra L/I, $\overline{a} \neq 0$, but $h(\overline{a}) = 0$ for any two-valued homomorphism h on L/I, a contradiction.

Proposition 3.7. Let I be an ideal of an orthomodular lattice L. The following statements are equivalent:

1. L is an I-quasi-Boolean algebra.
2. I is semiprime.
3. L is a strongly I-quasi-Boolean algebra.

Proof. (1) ⇒ (2): According to Definition 2.3(4), if L is a quasi-Boolean algebra, there is a Boolean algebra B and a lattice homomorphism $f: L \to B$ such that $f^{-1}(0) \subseteq I$. For every $a \in L$ we have

$$f(0) = f(a \wedge a') = f(a) \wedge f(a') = 0$$

$$f(1) = f(a \vee a') = f(a) \vee f(a') = 1$$

since a lattice homomorphism is order-preserving. Since a Boolean algebra can be characterized as a uniquely complemented ortholattice, we get $f(a') = f(a)'$, hence f preserves orthocomplements. Therefore $I_c \subseteq f^{-1}(0) \subseteq I$. By Proposition 3.4, I is semiprime.

(2) ⇒ (3): A semiprime ideal in an OML is the intersection of all prime ideals that contain it. By Definition 2.4(1), L is a strongly I-quasi-Boolean algebra.

(3) ⇒ (1): Follows directly from the definitions. ■

Recall that a *subalgebra* of an OL L is a subset $M \subseteq L$ such that (i) $a \in M \Rightarrow a' \in M$, (ii) $a, b \in M \Rightarrow a \vee b \in M$. A subalgebra M of an OL is an OL with the operations inherited from L. A subalgebra of L is a Boolean subalgebra if, with operations inherited from L, it is a Boolean algebra.

Example 3.8. The following example is given in D'Andrea and Pulmanová (1995). Let L be an infinite-dimensional, irreducible, complete, atomic OML with covering property. [As a concrete example we may consider the OML $L = L(H)$, the lattice of all closed linear subspaces of an infinite-dimensional Hilbert space.] Let F denote the set of all finite-dimensional elements of L. Then F is an orthomodular ideal in L which is contained in any other nonzero orthomodular ideal of L (D'Andrea and Pulmannová, 1995). Let $\Phi: L \to L/F$ be the canonical epimorphism.

Let B be any Boolean subalgebra of L. Put $L_1 = \{\Phi^{-1}(b): b \in \Phi(B)\}$. Now, L_1 is a subalgebra of L containing $F = \Phi^{-1}(0)$, and F is contained in every nonzero ideal of L_1. Moreover, $L_1/F \sim B$ is a Boolean algebra. Consequently, L_1 is an irreducible OML which is (strongly) I-quasi-Boolean with respect to every nonzero orthomodular ideal of L_1.

The method used in the above example will be used in the sequel for the characterization and construction of maximal quasi-Boolean subalgebras.

Let L be an OL and I an ideal of L such that I is the kernel of a homomorphism $h: L \to L/I$. Let B be a maximal Boolean subalgebra of L/I. Put

$$L_B = \{x \in L: x \in b \text{ for some } b \in B\}$$

It is easy to check that L_B is a subalgebra of L which contains I as an ideal. Moreover, $L_B/I \simeq B$, hence L_B is quasi-Boolean.

We claim that L_B is a maximal I-quasi-Boolean subalgebra of L. Assume that $L_1 \supseteq L_B$ is an I-quasi-Boolean subalgebra of L. Then L_1/I is a Boolean subalgebra of L/I containing B. Since B is maximal in L/I, we must have $L_1/I = B$. If there is $a \in L_1$, $a \notin L_B$, then $a/I \notin B$, $a/I \in L_1/I$, a contradiction.

Conversely, let L_1 be a subalgebra of L containing I which is maximal with the property of being I-quasi-Boolean. Then L_1/I is a Boolean subalgebra of L/I, which is contained in a maximal one, say B. Then $B \supset L_1/I \Rightarrow L_B \supset L_1$. Since L_1 is maximal with the property of being I-quasi-Boolean, we get $L_B = L_1$.

A subalgebra of an OML (as an ortholattice) is also an OML. Therefore, if I is an orthomodular ideal in an OML L, then L_B is a maximal sub-OML of L with the property of being I-quasi-Boolean (equivalently, that I is a semiprime ideal in L_B).

Summarizing, we have proved the following statement:

Theorem 3.9. Let I be an ideal of an OL L which is a kernel of a homomorphism. A subalgebra L_0 of L is maximal with the properties $L_0 \supseteq I$ and L_0/I is a Boolean algebra if and only if $L_0 = L_B$ for some maximal Boolean subalgebra B of L/I.

4. STATES ON ORTHOLATTICES

A *state* (finitely additive) on an OL L is a map m: $L \to [0, 1]$ such that $m(1) = 1$ and $a \perp b \Rightarrow m(a \vee b) = m(a) + m(b)$.

A state m is order-preserving: assume $a \leq b$; then $a \perp b'$ implies that $m(a \vee b') = m(a) + m(b') = m(a) + 1 - m(b)$, and from this we get $m(b) = m(a) + m(a' \wedge b)$. In other words, the orthomodular identity is satisfied in any state m on an ortholattice.

A state m on an OL L is called *Jauch–Piron* if

$$m(a) = 0, \quad m(b) = 0 \Rightarrow m(a \vee b) = 0$$

Equivalently, a state m is Jauch–Piron if the null set $m^{-1}(0)$ of m is an ideal in L.

A state m on an OL L is called a *p-state* (or an orthomodular state) if

$$m(a \vee b) = m(b) \qquad \text{whenever} \quad m(a) = 0$$

Equivalently, a state m is a p-state if $m^{-1}(0)$ is an orthomodular ideal.

A state m on an OL L is called *subadditive* if

$$m(a \vee b) \leq m(a) + m(b)$$

A state m on an OL L is called *valuation* if

$$m(a \vee b) + m(a \wedge b) = m(a) + m(b)$$

A state m on an OL L is called *two-valued* if

$$m(a) \in \{0, 1\} \qquad \text{for all} \quad a \in L$$

It is easy to see that the following inclusions hold:

valuation $\Rightarrow$ subadditive $\Rightarrow$ p-state $\Rightarrow$ Jauch–Piron state

If dim $H \geq 3$, then every completely additive state on $L(H)$ is Jauch–Piron; in particular, if H is finite dimensional, every state is Jauch–Piron. It is easy to find states on $L(H)$ of two-dimensional Hilbert space which are not Jauch–Piron. There are examples of Jauch–Piron states which are not orthomodular, and orthomodular states which are not subadditive (see, e.g., Pulmannová, 1994). A two-valued state is a valuation if and only if it is Jauch–Piron. On an OML, a state m is a valuation if and only if it is subadditive. On an ortholattice L, the following statement can be proved:

Proposition 4.1. A subadditive state m on an ortholattice L is a valuation if and only if $m^{-1}(1)$ is an orthomodular deductive system.

Proof. If m is a valuation, then the relation $a \equiv b$ iff $m(a \vee b) - m(a \wedge b) = 0$ is a congruence of L with the kernel $m^{-1}(0)$ and the quotient is a modular ortholattice (see, e.g., Birkhoff, 1973). According to Proposition 3.5(2), $m^{-1}(1)$ is an orthomodular deductive system.

Conversely, if m is subadditive and $m^{-1}(1)$ is an orthomodular deductive system, then $m^{-1}(0)$ is the kernel of a homomorphism of L onto an orthomodular lattice L_0. Let $\overline{a} \in L_0$ denote the equivalence class containing $a \in L$. Then $\overline{m}(\overline{a}) = m(a)$ defines a state on L_0. Subadditivity of m implies that $\overline{m}$ is subadditive, too, and hence it is a valuation. This implies that m is also a valuation. ■

The answer to the following question is not known to the authors:

Problem. Does there exist a subadditive state on an ortholattice which is not a valuation?

As a consequence of Theorem 3.9, we obtain the following:

Proposition 4.2. Let m be an orthomodular state on an OML L. A subalgebra L_0 of L is maximal with respect to the properties that $I = m^{-1}(0) \subset L_0$ and L_0/I is a Boolean algebra iff $L_0 = L_B$ for some maximal Boolean subalgebra of L/I.

Let L be an ortholattice and m a state on L. We say that m satisfies *Bell inequalities* [of type 2 (Pulmannová and Majerník, 1992) or of order 3 (Dvurečenskij and Länger, 1995b)] if, for any $a, b, c \in L$,

$$m(a) + m(b) + m(c) - m(a \wedge b) - m(a \wedge c)$$
$$- m(b \wedge c) + m(a \wedge b \wedge c) \leq 1 \qquad (1)$$

It was proved in that if m is a state on an OML L, then Bell inequalities (1) are satisfied if and only if $m(\overline{com}(a, b)) = 0$ for all $a, b \in L$. This yields the following statement:

Theorem 4.3. Let m be a state on an OML L and $I = m^{-1}(0)$. The following statements are equivalent:

1. Inequalities (1) are satisfied in m.
2. I is a semiprime ideal of L.
3. L is a I-quasi-Boolean algebra.

Recall that two elements a, b in an OML L are *compatible* if $\overline{com}(a, b) = 0$. We will write aCb if a, b are compatible. For any subset A of L, define $C(A) = \{b \in L: aCb, \forall a \in A\}$. The set $C(A)$ is called the *commutant* of A in L. It is well known that $C(A)$ is a subalgebra of L for any subset A of L. The set $C(L)$ is the center of L.

If p is an atom in an OML L, then $C(p)$ $[\equiv C(\{p\})] = [0, p'] \cup [p, 1]$. Indeed, bCp implies $p = b \wedge b \vee p \wedge b'$, and since p is an atom, either $p \wedge b = 0$, in which case $p \leq b'$, or $p \wedge b' = 0$, in which case $p \leq b$.

It is a well-known fact that an interval $[0, a]$ in an OML is an orthomodular ideal iff a belongs to the center $C(L)$ of L. Therefore, for every $a \in L$, $L_0 = C(a)$ is the greatest subalgebra L_0 of L with the property that $[0, a]$ is an orthomodular ideal in L_0. Combining this with our previous results, we obtain the following statement.

Theorem 4.4. Let L be an OML. For any $a \in L$, a subalgebra L_1 of L is maximal with respect to the property that L_1 is I-quasi-Boolean algebra, where $I = [0, a]$ if and only if $L_1 = L_B$ for some maximal Boolean subalgebra B of $C(a)/I$.

The following definition can be found in Pták and Pulmannová (1991). We say that a subset M of an OML L is *partially compatible with respect to* a (p.c. a in short, $a \in L$) if the following two conditions are satisfied:

1. $M \subset C(a)$.
2. $M \wedge a = \{m \wedge a: m \in M\}$ is a pairwise compatible set.

In particular, if a is an atom of L, then for any $b \in C(a)$, $b \wedge a$ is either 0 or a, hence the whole $C(a)$ is p.c. a. In the next proposition, we list some basic properties of partial compatibility (Pták and Pulmannová, 1991).

Proposition 4.5. Let L be an OML, M a subset of L, and a, a_i ($i \in I$) elements of L. The following are true:

1. M is p.c. a iff $\{x, y\}$ p.c. a for all $x, y \in M$.
2. $\{x, y\}$ p.c. a iff $\{x, y\} \subset C(a)$ and $a \leq \underline{com}\{x, y\}$. In particular, $\{x, y\}$ p.c. $\underline{com}\{x, y\}$.
3. $\{x, y\}$ p.c. a_i for all $i \in I$ implies $\{x, y\}$ p.c. $\vee_i a_i$ and $\wedge a_i$.
4. A maximal p.c. a set Q containing a given set M (by Zorn's lemma, it exists) is a subalgebra of L.

Proposition 4.6. A maximal p.c. a set Q in an OML L ($a \in L$) coincides with a subalgebra L_0 of L which is maximal with respect to the properties that $[0, a']$ is an orthomodular ideal in L_0 and L_0 is $[0, a']$-quasi-Boolean.

Proof. (1) Let Q be a maximal p.c. a subset of L. We then have the following properties: $Q \subset C(a)$ and $[0, a']$ is an orthomodular ideal in $C(a)$. Therefore $C(a)$ can be factorized to $C(a) = [0, a] \times [0, a']$ and the quotient $C(a)/[0, a']$ is isomorphic with the interval $[0, a]$. Moreover, $Q/[0, a']$ is isomorphic with $Q \wedge a$. According to Proposition 4.5(4), it is a subalgebra of $[0, a]$, and since it is pairwise compatible, it is a Boolean algebra. Finally, $Q \wedge a$ is a maximal Boolean subalgebra of $[0, a]$, because Q is a maximal p.c. a set. Observe that $Q = L_B$, where $B = Q/[0, a'] = Q \wedge a$.

2. Let L_0 be a subalgebra of L which is maximal with the property that L_0 is a $[0, a']$-quasi-Boolean algebra. Then $[0, a']$ must be an orthomodular ideal of L_0, and hence $L_0 \subset C(a)$. In addition, $L_0/[0, a']$ is a Boolean algebra. Since $L_0/[0, a']$ is isomorphic with $L_0 \wedge a$, this yields that L_0 is p.c. a. If L_0 is not maximal p.c. a subset of L, then there is a maximal p.c. a subset Q of L containing L_0. According to Proposition 4.5(4), Q is a subalgebra of L. Then $L_0/[0, a'] \subset Q/[0, a']$, and $Q/[0, a']$ is a Boolean algebra. Hence Q is $[0, a']$-quasi-Boolean algebra. Owing to maximality of L_0, we get $L_0 = Q$. ∎

In particular, if p is an atom of L, then $C(p) = [0, p'] \cup [p, 1]$, and $C(p)/[0, p'] \equiv [0, p]$ is isomorphic with the Boolean algebra 2. Hence $C(p)$ itself is the greatest p.c. p subalgebra of L. Equivalently, $[0, p']$ is a prime ideal of $C(p)$, and $C(p)$ is the greatest $[0, p']$-quasi-Boolean subalgebra of L.

Let m be a state on an ortholattice L. If there is an element $a \in L$ such that $m(b) = 0$ if and only if $b \perp a$, we say that a is the *support* of m, and write $a = s(m)$.

On the Hilbert space logic $L(H)$, dim $H \geq 3$, every completely additive state m has a support, namely if m corresponds to a density operator D with the spectral decomposition $D = \Sigma_i w_i P_i$, then its support is $s(m) = \vee P_i$.

We say that a subset M of an OML L has a *joint distribution* in a state m if for every finite subset $\{a_1, a_2, \ldots, a_n\}$ of M there is a probability measure μ on $\mathcal{B}(\mathbb{R}^n)$ such that

$$\mu(E_1 \times E_2 \times \cdots \times E_n) = m(x_{a_1}(E_1) \wedge \cdots \wedge x_{a_n}(E_n))$$

where x_{a_i} is the proposition observable corresponding to the element a_i [that is, $x_{a_i}(E)$, $E \in \mathcal{B}(\mathbb{R})$, is equal to a_i, a_i', 1, or 0 if E contains 1, 0, both 1 and 0, or none of 1 and 0, respectively]. For the following facts about joint distribution see, e.g., Pulmannová (1985), Pták and Pulmannová (1991), and Pulmannová and Dvurečenskij (1985). A subset M of L has a joint distribution in a state m if and only if $m(\underline{com}(a_1, \ldots, a_n)) = 1$ for any $a_1, \ldots, a_n \in M$. If $\wedge_{F \subset M} \underline{com} F$ exists, where the infimum is taken over all finite subsets F of M, we call it the (lower) commutator of the set M, and M p.c. $\underline{com} M$ whenever $\underline{com} M$ exists. In particular, if L is a complete lattice, then $\underline{com} M$ exists for any $M \subset L$. In a separable Hilbert-space logic $L(H)$, the condition that M has a joint distribution in a σ-additive state m can be rewritten as $m(\underline{com} M) = 1$. If $L(M)$ denotes a subalgebra of L generated by M, then $\underline{com} L(M) = \underline{com} M$. Moreover, since $L(M)$ p.c. $\underline{com} M$, $L(M) \wedge \underline{com} M$ is a Boolean subalgebra of the OML $(\underline{com} M]$ (or, equivalently, of the quotient $C(\underline{com} M)/[0, \overline{\underline{com}} M]$, and for any $a \in C(\underline{com} M)$, $a = a \wedge \underline{com} M \vee a \wedge \overline{\underline{com}} M$. Therefore $m(a) = m(a \wedge \underline{com} M)$, whenever $m(\underline{com} M) = 1$.

This means that a maximal subset of $L(H)$ having a joint distribution in a state m is a subalgebra L_0 of $L(H)$ maximal with the property $m(\underline{com} L_0) = 1$. Moreover, elements of L_0 can be treated, with regard to their stochastic properties in the state m, as the classical propositions $L_0 \wedge \underline{com} L_0$.

Now if the support $s(m)$ of the state m exists, then $m(\underline{com}(a_1, \ldots, a_n)) = 1$ iff $s(m) \leq \underline{com}(a_1, \ldots, a_n)$. Therefore, a maximal p.c. $s(m)$ set can also be characterized as a subalgebra L_0 of L which is maximal with respect to the properties $L_0 \subset C(s(m))$ and L_0 have a joint distribution in the state m. For any $a \in C(s(m))$ we have $a = a \wedge s(m) \vee a \wedge s(m)'$, hence $m(a) = m(a \wedge s(m))$. If $s(m)$ is an atom, then $a \wedge s(m)$ is either 0 or $s(m)$, whence $m(a)$ is either 0 or 1. These remarks will be used in the next section.

5. SIMULTANEOUSLY DEFINITE PROPERTIES

Consider a quantum system represented by a Hilbert space H whose state is represented at some moment by some density (positive, Hermitian, trace-class one) operator D on H. Each projection operator P on H defines a proposition of the system and has eigenvalues 1 and 0, which can be interpreted as "true" and "false." There is a well-known bijection between the set of projections and the set $L(H)$ of closed subspaces of H. The corresponding state is then given by the mapping $P \mapsto \mathrm{Tr}(PD)$, $P \in L(H)$. The following question now arises. Of the propositions of the system, represented by $L(H)$, which can be regarded as actually having simultaneously well-defined values (0 or 1) in state D? Three proposals are considered in Bell and Clifton (1995). We use the symbol $L(D)$ for the set of propositions in question.

I. The orthodox proposal advocated by von Neumann (1955) and Dirac (1958) is

$$L(D) = \{P \in L(H)\colon \mathrm{Tr}(PD) = 1 \text{ or } 0\}$$

Equivalently,

$$L(D) = \{P \in L(H)\colon s(D) \leq P \text{ or } s(D) \leq P'\}$$

where $s(D)$ denotes the support of the state D.

If $s(D)$ is an atom (i.e., if D is a pure state), then $L(D)$ coincides with $C(s(D))$, the maximal p.c. $s(D)$ subalgebra of $L(H)$. According to Proposition 4.6, $L(D)$ is the maximal $[0, s(D)']$-quasi-Boolean subalgebra of L. Clearly, $s(D)$ is a central atom of $L(D)$, and if H is finite-dimensional, then $L(D)$ is easily seen to be generated by $s(D)$ and all atoms contained in $s(D)'$.

II. We require that $L(D)$ consists of propositions that have definite values in all pure states corresponding to the spectral projections of D with nonzero eigenvalues

$$L(D) = \{P \in L(H)\colon \forall \underline{P} \in P_D, \underline{P} \leq P \text{ or } \underline{P} \leq P'\}$$

where P_D denotes the set of spectral projections of D with nonzero eigenvalues.

Writing $L_0 = \bigcap_{\underline{P} \in P_D} C(\underline{P})$, $I = [0, s(D)']$, and taking into account that $s(D) = \vee \underline{P}$, it can be easily checked that $I \subset L(D) \subset L_0$, $L(D)$ is I-quasi-Boolean, and $L(D) = L_B$, where B is the Boolean subalgebra of L_0/I generated by the classes corresponding to the elements $\underline{P} \in P_D$ and $s(D)'$. $L(D)$ is a maximal I-quasi-Boolean subalgebra of L with the additional property that $\underline{P} \in P_D$ are its central atoms.

III. The proposal for $L(D)$ due to Bub and Clifton (1995) is as follows. Let $\{R_i\}$ be the (finite, for simplicity) set of spectral projections of some observable represented by the self-adjoint operator R, and (for pure D) define $D_{R_i} = (D \vee R_i') \wedge R_i$ for all i. Let $\{D_{R_j}\}$ be the set of all nonzero D_{R_i}. The proposal for $L(D)$ is

$$L(D) = \{P \in L(H)\colon \forall j, D_{R_j} \leq P \text{ or } D_{R_j} \leq P'\}$$

Every D_{R_j} can be interpreted as the Lüders change of the state D after measurement of R_j. Correspondingly, the state D after a measurement of R

is changed into a mixture of the states D_{R_j}. The new state has the support $\vee D_{R_j}$, hence the ideal corresponding to zero-valued propositions in this new state is $I = [0, (\vee D_{R_j})']$. With this interpretation, **III** is analogous to **II**, but takes into account the Lüders change of the initial state after a measurement of a chosen observable.

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